The method of expansion of Feynman integrals

S.A. Larin

Institute for Nuclear Research of the Russian Academy of Sciences,
60th October Anniversary Prospect 7a, Moscow 117312, Russia

Abstract

The method of expansion of integrals in external parameters is suggested. It is quite universal and works for Feynman integrals both in Euclidean and Minkowski regions of momenta.

During the last two decades different techniques were developed [1]-[10] for asymptotic expansions of Feynman integrals in quantum field theory. These techniques allow to perform practical calculations when exact integrations are not possible. In the present paper we suggest the new effective method of expansion of integrals in external parameters – 'the method of cancelling factors'. It is quite universal and works for Feynman integrals both in Euclidean and Minkowski regions of momenta.

To demonstrate the essence of the method of cancelling factors we begin with the following simple integral

$$\int_0^1 \frac{dx}{(x+t)(1+x)} = \frac{\ln(1+t)-\ln(t)-\ln(2)}{1-t}. \tag{1}$$

We want to expand this integral in small external parameter t before the integration. The naive expansion of the integrand in the Taylor series in t does not work since it produces the non-integrable singularities at $x=0$.

To get the correct expansion let us distinguish two factors in the integrand : the expanding factor $\frac{1}{x+t}$ (the expansion of this factor initiates the expansion of the whole integral) and the cancelling factor $\frac{1}{1+x}$ (this factor is used to cancel singularities arising in the expansion of the expanding factor) . We subtract from and add to the cancelling factor its Taylor series in x up to some power n

$$\int_0^1 \frac{dx}{(x+t)(1+x)} = \int_0^1 dx \frac{1}{x+t}\left[\frac{1}{1+x} - \sum_{j=0}^{n}(-x)^j\right] + \int_0^1 dx \frac{1}{x+t}\sum_{j=0}^{n}(-x)^j. \tag{2}$$

In the first integral of the right hand sight of eq.(2) we can now safely perform the Taylor expansion of the expanding factor $\frac{1}{x+t}$ in t up to and including the order t^n. This will not generate anymore non-integrable singularities at $x=0$ since the factor in the square brackets (the subtracted cancelling factor) has the behavior $O(x^{n+1})$ and thus suppresses singularities arising in the expansion of $\frac{1}{x+t}$.

Finally we get the desired expansion

$$\int_0^1 \frac{dx}{(x+t)(1+x)} = \int_0^1 dx \sum_{k=0}^{n}\frac{(-t)^k}{x^{k+1}}\left[\frac{1}{1+x} - \sum_{j=0}^{n}(-x)^j\right] + \int_0^1 dx \frac{1}{x+t}\sum_{j=0}^{n}(-x)^j + O(t^{n+1}\ln t). \tag{3}$$

In each term of this expression some factor is expanded and integrations reproduce the expansion in t of the exact result in the right hand side of eq.(1).

Let us generalize the above considerations. For a given integral the method of cancelling factors distinguishes the expanding factor (which will be expanded in external parameters of the integral) and several (one or more) cancelling factors (which will be used to cancel singularities in the expansion of the expanding factor, the number of cancelling factors is determined by the necessity to suppress all arising singularities). For each cancelling factor one adds and subtracts its expansion in the integration variable up to the necessary order at some singular point (the point where singularities appear in the expansion of the expanding factor).

At last in the term containing the product of subtracted cancelling factors one expands the expanding factor in external parameters up to the necessary order without generating non-integrable singularities.

Let us now apply the method of cancelling factors to Feynman integrals in quantum field theory. To regularize divergent integrals we will use dimensional regularization [11] for convenience (but the method is regularization independent). The dimension of the momentum space is defined as $D = 4-2\epsilon$ where ϵ is the parameter defining the deviation of the dimension from its physical value 4.

We consider first the expansion at large external momentum squared q^2 of the following one-loop Feynman integral of the propagator type

$$\int \frac{d^D k}{(k^2 - m_1^2 + i0)[(k+q)^2 - m_2^2 + i0]}, \tag{4}$$

where k is the integration momentum, m_1 and m_2 are the masses of the propagators. Below in the paper we will omit the 'causal' $i0$ for brevity.

The expansion in large q^2 means alternatively the expansion in small m_1 and m_2. We can not just naively expand the integrand in Taylor series in m_1^2 and m_2^2. Such an expansion produces incorrect result since it generates infrared singularities at $k = 0$ and $k + q = 0$. (At first glance the singularities are generated at $k^2 = 0$ and $(k+q)^2 = 0$ but it is known that Feynman integrals with one external momentum can be always treated in Euclidean region of momenta where conditions $k^2 = 0$ and $k = 0$ are equivalent.) To get the correct expansion let us distinguish two factors in the integrand : the expanding factor $\frac{1}{k^2 - m_1^2}$ and the cancelling factor $\frac{1}{(k+q)^2 - m_2^2}$. We subtract from and add to the cancelling factor its Taylor series in k:

$$\int \frac{d^D k}{(k^2 - m_1^2)[(k+q)^2 - m_2^2]} = \int d^D k \frac{1}{k^2 - m_1^2}\left[\frac{1}{(k+q)^2 - m_2^2} - T_k^{2n_1}\frac{1}{(k+q)^2 - m_2^2}\right] + \tag{5}$$
$$\int d^D k \frac{1}{k^2 - m_1^2} T_k^{2n_1}\frac{1}{(k+q)^2 - m_2^2},$$

where

$$T_k^{2n_1}\frac{1}{(k+q)^2 - m_2^2} = \sum_{j=0}^{2n_1} \frac{\partial^j}{\partial k^{\mu_1} \ldots k^{\mu_j}} \frac{1}{(k+q)^2 - m_2^2}|_{k=0} \frac{k^{\mu_1}\ldots k^{\mu_j}}{j!}$$

is the Taylor expansion of the cancelling factor in k up to some order $2n_1$.

In the first integral of the right hand sight of eq.(5) we can now perform the Taylor expansion in m_1^2 of the expanding factor $\frac{1}{k^2 - m_1^2}$ up to and including the order $(m_1^2)^{n_1+1}$. This expansion will not generate anymore infrared singularities since the factor in the square brackets (the subtracted cancelling factor) behaves as $O(k^{2n_1+1})$ and thus suppresses the infrared singularities at $k = 0$ arising in the Taylor expansion

$$T_{m_1^2}^{n_1+1}\frac{1}{(k^2 - m_1^2)} = \sum_{j=0}^{n_1+1} \frac{(m_1^2)^j}{(k^2)^{j+1}}.$$

Thus we get

$$\int \frac{d^D k}{(k^2+m_1^2)[(k+q)^2-m_2^2]} = \int d^D k\, T^{n_1+1}_{m_1^2} \frac{1}{k^2-m_1^2}\frac{1}{(k+q)^2-m_2^2} + \int d^D k \frac{1}{k^2-m_1^2} T^{2n_1}_k \frac{1}{(k+q)^2-m_2^2} + O\left((m_1^2)^{n_1+2}\right), \tag{6}$$

where we took into account that the term containing both Taylor expansions $T^{n_1+1}_{m_1^2}$ and $T^{2n_1}_k$ is zero due to the known property of the dimensional regularization to nullify the integrals without external parameters (massless tadpoles). The approximation here $O\left((m_1^2)^{n_1+2}\right)$ and approximations below in the paper are written up to logarithms.

This is already a kind of expansion but we can continue further with the expansion of the first term in the right hand side of eq.(6). For this purpose it is convenient to make in this term the shift of integration momentum $k \to k-q$, so we get

$$\int \frac{d^D k}{(k^2+m_1^2)[(k+q)^2-m_2^2]} = \int d^D k \frac{1}{k^2-m_2^2} T^{n_1+1}_{m_1^2} \frac{1}{(k-q)^2-m_1^2} + \int d^D k \frac{1}{k^2-m_1^2} T^{2n_1}_k \frac{1}{(k+q)^2-m_2^2} + O\left((m_1^2)^{n_1+2}\right). \tag{7}$$

Then in the first term the factor $\frac{1}{k^2-m_2^2}$ is considered as the expanding factor and the factor $T^{n_1+1}_{m_1^2}\frac{1}{(k-q)^2-m_1^2}$ as the cancelling factor. Again we subtract from and add to the cancelling factor its Taylor expansion $T^{2n_2}_k$ in k. Then in the term containing the subtracted cancelling factor we can make the Taylor expansion $T^{n_2+1}_{m_2^2}$ of the expanding factor $\frac{1}{k^2-m_2^2}$ (in the same way as the expansion in m_1^2 during the derivation of eq.(6)). Finally we come to the expansion (after nullification of massless tadpoles)

$$\int \frac{d^D k}{(k^2+m_1^2)[(k+q)^2-m_2^2]} = \int d^D k\, T^{n_2+1}_{m_2^2} \frac{1}{k^2-m_2^2} T^{n_1+1}_{m_1^2} \frac{1}{(k-q)^2-m_1^2} + \int d^D k \frac{1}{k^2-m_2^2} T^{2n_2}_k T^{n_1+1}_{m_1^2} \frac{1}{(k-q)^2-m_1^2} + \int d^D k \frac{1}{(k^2-m_1^2)} T^{n_2+1}_{m_2^2} T^{2n_1}_k \frac{1}{(k+q)^2-m_2^2} + O\left((m_1^2)^{n_1+2}, (m_2^2)^{n_2+2}\right), \tag{8}$$

where in the last term (which is nothing but the last term in eq.(7)) we applied the Taylor expansion in m_2 which does not effect integrations. This result agrees with the recipe explicitly formulated in [7] for the large q^2 expansion of propagator integrals. The new point here is the simple derivation of the expansion.

As the next application of the method we shall consider the Sudakov formfactor [12] which is a typically Minkowskian case not reducible to the Euclidean space of momenta. The corresponding one-loop Feynman integral is

$$\int \frac{d^D k}{(k^2-m^2)(k^2-2p_1k)(k^2-2p_2k)}, \tag{9}$$

where external momenta are on mass shell: $p_1^2 = p_2^2 = 0$. The integral will be expanded in small mass m^2 which means the expansion in terms of the ratio $\frac{m^2}{q^2}$ where $q = p1 - p2$. This expansion was obtained in [9]. Here we give the simple derivation of the expansion with the method of cancelling factor.

The expansion of the integrand in m^2 generates infrared singularities at $k^2 = 0$. We distinguish here the expanding factor $\frac{1}{k^2-m^2}$ and two cancelling factors $\frac{1}{k^2-2p_1k}$ and $\frac{1}{k^2-2p_2k}$. For each cancelling factor we subtract and add its expansion in k^2

$$T^n_{k^2}\frac{1}{k^2-2p_ik} = -\sum_{j=0}^{n}\frac{(k^2)^j}{(2p_ik)^{j+1}},\ i=1,2.$$

In this way we get

$$\int\frac{d^Dk}{(k^2-m^2)(k^2-2p_1k)(k^2-2p_2k)} = \tag{10}$$
$$\int d^Dk\frac{1}{k^2-m^2}\left[(1-T^n_{k^2}+T^n_{k^2})\frac{1}{k^2-2p_1k}\right]\left[(1-T^n_{k^2}+T^n_{k^2})\frac{1}{k^2-2p_2k}\right] =$$
$$\int d^Dk\frac{1}{k^2-m^2}\left[(1-T^n_{k^2})\frac{1}{k^2-2p_1k}\right]\left[(1-T^n_{k^2})\frac{1}{k^2-2p_2k}\right] +$$
$$\int d^Dk\frac{1}{k^2-m^2}\frac{1}{k^2-2p_2k}T^n_{k^2}\frac{1}{k^2-2p_1k} + \int d^Dk\frac{1}{k^2-m^2}\frac{1}{k^2-2p_1k}T^n_{k^2}\frac{1}{k^2-2p_2k},$$

where in the last equation we took into account that the terms containing two factors with $T^n_{k^2}$ are zero. (Here is a technical subtlety. Dimensional regularization does not regularize individual terms in the last equation although it regularizes the original integral (9). Strictly speaking, we should introduce analytic regularization $\frac{1}{k^2-2p_ik} \to \frac{1}{(k^2-2p_ik)^{1+\lambda_i}}$, $i = 1, 2$ in eq.(9) in addition to dimensional regularization, where λ_i are the arbitrary parameters of analytic regularization. But this technical subtlety does not change the derivation of the expansion and the final result.)

Then in the first term of the last equation we can expand the expanding factor as

$$T^n_{m^2}\frac{1}{k^2-m^2} = \sum_{j=0}^{n}\frac{(m^2)^j}{(k^2)^{j+1}}$$

without generating infrared singularities at $k^2 = 0$. This is because the first square bracket (the first subtracted cancelling factor) behaves as $O\left(\frac{(k^2)^{n+1}}{(2p_1k)^{n+2}}\right)$ and the second square bracket (the second subtracted cancelling factor) behaves as $O\left(\frac{(k^2)^{n+1}}{(2p_2k)^{n+2}}\right)$ at small k^2. The scalar product $2p_1k$ can be small simultaneously with k^2 and then the first square bracket does not suppress infrared singularities at small k^2. In this case the second square bracket ensures the suppression of infrared singularities at $k^2 = 0$ and vice versa. (The scalar products $2p_1k$ and $2p_2k$ are not simultaneously small at small k^2 since the momenta p_1 and p_2 are different).

That is why we need two cancelling factors here. Finally we get the following expansion (after taking into account that terms containing two factors with Taylor expansions are zero)

$$\int \frac{d^D k}{(k^2-m^2)(k^2-2p_1k)(k^2-2p_2k)} = \int d^D k\, T^n_{m^2} \frac{1}{k^2-m^2} \frac{1}{k^2-2p_1k} \frac{1}{k^2-2p_2k} \tag{11}$$
$$+ \int d^D k \frac{1}{k^2-m^2} \frac{1}{k^2-2p_2k} T^n_{k^2} \frac{1}{k^2-2p_1k}$$
$$+ \int d^D k \frac{1}{k^2-m^2} \frac{1}{k^2-2p_1k} T^n_{k^2} \frac{1}{k^2-2p_2k} + O\left((m^2)^{n+1}\right).$$

Here the second and third integrals are not individually regularized by dimensional regularization as was already mentioned above but their sum is regularized.

To conclude, in the present paper we described the method of cancelling factors for expansion of integrals in external parameters, giving three examples of its applications.

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